ON A CLASS OF EXACT PARTICULAR SOLUTIONS OF THE EQUATIONS OF TRANSONIC GAS FLOWS

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The paper presents exact particular solutions of the equations of transonic gas flows, analogous to the solutions derived in [1–3] for the case of short waves. These solutions are used to construct the flow around a body in a supersonic stream with an attached shock.

§1. Let u, v be the components of the velocity vector in the directions x, y of a rectangular system of coordinates, a the speed of sound, t the time, P_*, ρ_*, a_* the critical parameters of the flow, l a characteristic length, and ε a small parameter. We introduce the following notation:

$$\frac{u}{a_*} = 1 - \varepsilon\frac{U}{\varkappa+1}, \qquad \frac{v}{a_*} = \varepsilon^{3/2}\frac{V}{\varkappa+1}, \qquad \frac{x}{a_* t_*} = \varepsilon X,$$

$$\frac{y}{a_* t_*} = \varepsilon^{1/2} Y, \quad a_*^2 = \frac{p_*\varkappa}{\rho_*}, \quad \tau = \frac{t}{2t_*}, \quad t_* = \frac{l}{\varepsilon a_*}.$$

The equations of unsteady transonic flow [4, 5] are then

$$\frac{\partial U}{\partial \tau} - U\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \quad \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} = 0. \tag{1.1}$$

We shall seek particular solutions of (1.1) in the form

$$U = \varphi_2(q,\tau)Y^2 + \varphi_1(q,\tau)Y + \varphi_0(q,\tau)$$

$$V = \psi_3(q,\tau)Y^3 + \psi_2(q,\tau)Y^2 + \psi_1(q,\tau)Y + \psi_0(q,\tau)$$

$$X = qY^2 + \chi_1(q,\tau)Y + \chi_0(q,\tau) \quad (q - \text{parameter}). \tag{1.2}$$

Substituting (1.2) into (1.1), we obtain a system of first-order partial differential equations for the functions φ_2, φ_1, φ_0, ψ_3, ψ_2, ψ_1, ψ_0, χ_1, χ_0, where the subscripts denote differentiation:

$$\varphi_{2\tau} - \varphi_2\varphi_{2q} - 2q\psi_{3q} + 3\psi_3 = 0,$$

$$\varphi_{1\tau} + \varphi_{2\tau}\chi_{1q} - \varphi_{2q}\chi_{1\tau} - \varphi_1\varphi_{2q} - \varphi_2\varphi_{1q} +$$
$$+ 2\psi_2 + 3\psi_3\chi_{1q} - 2\psi_{2q}q - \chi_1\psi_{3q} = 0,$$

$$\varphi_{0\tau} + \varphi_{1\tau}\chi_{1q} + \varphi_{2\tau}\chi_{0q} - \varphi_{1q}\chi_{1\tau} - \varphi_{2q}\chi_{0\tau} - \varphi_{2q}\varphi_0 -$$
$$- \varphi_1\varphi_{1q} - \varphi_2\varphi_{0q} + \psi_1 + 2\psi_2\chi_{1q} + 3\psi_3\chi_{0q} -$$
$$- 2q\psi_{1q} - \chi_1\psi_{2q} = 0,$$

$$\varphi_{0\tau}\chi_{1q} + \varphi_{1\tau}\chi_{0q} - \varphi_{0q}\chi_{1\tau} - \varphi_{1q}\chi_{0\tau} - \varphi_0\varphi_{1q} - \varphi_1\varphi_{0q} + \tag{1.3}$$
$$+ \chi_{1q}\psi_1 + 2\psi_2\chi_{0q} - 2q\psi_{0q} - \chi_1\psi_{1q} = 0,$$

$$\varphi_{0\tau}\chi_{0q} - \varphi_{0q}\chi_{0\tau} - \varphi_0\varphi_{0q} + \psi_1\chi_{0q} - \chi_1\psi_{0q} = 0,$$

$$2\psi_2 - 2q\varphi_{2q} + \psi_{3q} = 0,$$

$$\varphi_1 + 2\varphi_2\chi_{1q} - 2q\varphi_{1q} - \varphi_{2q}\chi_1 + \psi_{2q} = 0,$$

$$\chi_{1q}\varphi_1 + 2\varphi_2\chi_{0q} - 2q\varphi_{0q} - \varphi_{1q}\chi_1 + \psi_{1q} = 0,$$

$$\varphi_1\chi_{0q} - \varphi_{0q}\chi_1 + \psi_{0q} = 0,$$

Following [1], we reduce the system (1.3) to the form

$$\varphi_{2\tau} + 4q\varphi_2 + 3\psi_3 - \varphi_2\varphi_{2q} - 4q^2\varphi_{2q} = 0, \tag{1.4}$$

$$\psi_1 = \varphi_{0q}(\varphi_2 + 4q^2) - \varphi_1\chi_1 - \varphi_{0\tau},$$

$$\psi_2 = {}^1/_2\,\varphi_{1q}(\varphi_2 + 4q^2) - \varphi_2\chi_1 - q\varphi_1 - {}^1/_2\,\varphi_{1\tau},$$

$$\psi_{0q} = \varphi_{0q}\chi_1 - \varphi_1\chi_{0q},$$

$$\chi_{1q}\varphi_2 + 4q^2\chi_{1q} - \chi_{1\tau} - \varphi_1 - 4q\chi_1 = 0, \qquad \text{(1.4) (cont'd)}$$

$$\psi_{2q} = \varphi_{2q}\chi_1 + 2q\varphi_{1q} - 2\varphi_2\chi_{1q} - \varphi_1,$$

$$\chi_{0q}\varphi_2 + 4q^2\chi_{0q} - \chi_{0\tau} - \varphi_0 - \chi_1^2 = 0,$$

$$\psi_{1q} = \varphi_{1q}\chi_1 + 2q\varphi_{0q} - 2\varphi_2\chi_{0q} - \varphi_1\chi_{1q}.$$

The function φ_2 are determined from the equation

$$\varphi_{2\tau q} - \varphi_{2qq}(\varphi_2 + 4q^2) - \varphi_{2q}(\varphi_{2q} - 2q) + 2\varphi_2 = 0 \tag{1.5}$$

which has the particular solutions

$$\varphi_2 = A(\tau)\sqrt{q + B(\tau)} - D(\tau)\,q - \Phi(\tau),$$

$$\varphi_2 = -q^2 + C(\tau).$$

Here $C(\tau)$ and $B(\tau)$ are arbitrary functions, and D and Φ depend on the choice of $B(\tau)$. In the steady case A, B, C = const., and the general solution of (1.5) and its singular integral are

$$\varphi_2 = 2A\sqrt{q + B} - 4Bq - 8B^2, \tag{1.6}$$

$$\varphi_2 = -q^2 + C.$$

Let us assume $\varphi_{0q} = \varphi_{2q}\nu + \mu$, $\chi_{0q} = \nu$, $\varphi_{1q} = \varphi_{2q} \times \chi_{1qq} = \eta$. Transformations analogous to those used in [1] reduce the system (1.4) to the form

$$\mu_q(\varphi_2 + 4q^2) + 2\mu(\varphi_{2q} + 3q) - \mu_\tau - 2\xi\chi_1 = 0,$$

$$\nu_q(\varphi_2 + 4q^2) + 8q\nu - \nu_\tau - \mu - 2\chi_1\chi_{1q} = 0,$$

$$\xi_q(\varphi_2 + 4q^2) + 2\xi(\varphi_{2q} + q) - \xi_\tau = 0,$$

$$\eta_q(\varphi_2 + 4q^2) + \eta(\varphi_{2q} + 12q) - \eta_\tau - \xi_q = 0,$$

$$\psi_0 = \int(\varphi_{0q}\chi_1 - \varphi_1\chi_{0q})\,dq, \tag{1.7}$$

$$\psi_2 = {}^1/_2\,\varphi_{1q}(\varphi_2 + 4q^2) - \varphi_2\chi_1 - q\varphi_1 - {}^1/_2\,\varphi_{1\tau},$$

$$\psi_1 = \varphi_{0q}(\varphi_2 + 4q^2) - \varphi_1\chi_1 - \varphi_{0\tau},$$

$$3\psi_3 = \varphi_{2q}(\varphi_2 + 4q^2) - 4q\varphi_2 - \varphi_{2\tau}.$$

To find solutions of the form (1.2), we must integrate equation (1.5) and solve consecutively the above four equations.

§2. In the steady case the general solution of (1.7) is

$$\xi = C\exp\left(-\int\frac{2(\varphi_2' + q)}{\varphi_2 + 4q^2}\,dq\right),$$

$$\eta = \exp\left(-\int\frac{\varphi_2' + 12q}{\varphi_2 + 4q^2}\,dq\right). \tag{2.1}$$

$$\cdot\left[\int\frac{\xi'}{\varphi_2+4q^2}\exp\left(\int\frac{\varphi_2'+12q}{\varphi_2+4q^2}dq\right)dq+C_1\right],$$

$$\mu=\exp\left(-\int\frac{2(\varphi_2'+3q)}{\varphi_2+4q^2}dq\right)\cdot$$

$$\cdot\left[\int\frac{2\xi\chi_1}{\varphi_2+4q^2}\exp\left(\int\frac{2(\varphi_2'+3q)}{\varphi_2+4q^2}dq\right)dq+C_2\right],\qquad (2.1)\ \text{(Cont'd)}$$

$$\nu=\exp\left(-\int\frac{8q}{\varphi_2+4q^2}dq\right)\cdot$$

$$\cdot\left[\int\frac{\mu+2\chi_1\chi_1'}{\varphi_2+4q^2}\exp\left(\int\frac{8q}{\varphi_2+4q^2}dq\right)dq+C_3\right].$$

When $\varphi_2 = -q^2 + C$, then for ξ and μ identically equal equal to zero the solutions (2.1) represent a Prandtl-Mayer type flow together with a flow of another type. The transition line—a straight line when $\nu \equiv 0$, $C = 0$, and a curved line when $\nu \neq 0$—always passes through the singular point. Thus, the system (2.1) describes both the subsonic and the supersonic parts of the flow.

For $\chi_1 = 0$, $\varphi_1 = 0$ the functions φ_0, χ_0, ψ_1, ψ_3 are determined from the equations

$$\varphi_0=\chi_0'(\varphi_2+4q^2),\qquad \psi_1=\varphi_0'(\varphi_2+4q^2),$$

$$\nu_q(\varphi_2+4q^2)+8q\nu-\mu=0$$

$$\mu_q(\varphi_2+4q^2)+2\mu(\varphi_2'+3q)=0,$$

$$\psi_3={}^1/_3[\varphi_2'(\varphi_2+4q^2)-4q\varphi_2]\quad (\chi_0=\textstyle\int\nu\,dq).$$

When $\varphi_2=2A\sqrt{q+B}-4Bq-8B^2$, the general solution of this system is

$$\mu=C|q+B|^{-5/4}|\varphi_2+4q^2|^{-1/3};$$

$$\nu=|q+B|^{1/4}|\varphi_2+4q^2|^{-4/3}[C_1\sqrt{q+B}-2C];$$

$$\varphi_0=|q+B|^{-5/4}|\varphi_2+$$

$$+4q^2|^{-1/3}[C_1\sqrt{q+B}-2C](\varphi_2+4q^2);$$

$$\psi_1=-\varphi_2\varphi_0-C|\varphi_2+4q^2|^{2/3}|q+B|^{-5/4};$$

$$\psi_3=-{}^2/_3[\varphi_2'(\varphi_2+4q^2)-2q\varphi_2].$$

§3. In terms of the coordinates

$$X=\frac{r}{a_*t_*\varepsilon},\quad Y=\frac{r}{a_*t_*}\varepsilon^{-1/2},\quad \tau=\frac{t}{2t_*},\quad t_*=\frac{l}{\varepsilon a_*}$$

the equations of axisymmetric flow are [5]

$$\frac{\partial U}{\partial\tau}-U\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}+\frac{V}{Y}=0,\quad \frac{\partial V}{\partial X}+\frac{\partial U}{\partial Y}=0.\qquad(3.1)$$

These equations have a class of exact particular solutions of the form

$$U=\varphi_2(q,\tau)Y^2+\int(\varphi_{2q}\nu+\mu)\,dq,\quad X=qY^2+\int\nu\,dq$$
$$V=\psi_3(q,\tau)Y^3+2Y\int[-\varphi_2\nu+q\varphi_{2q}\nu+q\mu]\,dq,\qquad(3.2)$$

where the functions φ_2, ν, μ are determined by the equations

$$\nu_\tau-\nu_q(\varphi_2+4q^2)-8q\nu+\mu=0,$$

$$\mu_\tau-\mu_q(\varphi_2+4q^2)-\mu(2\varphi_{2q}+4q)=0,$$

$$\varphi_{2\tau q}-\varphi_{2qq}(\varphi_2+4q^2)-\varphi_{2q}(\varphi_{2q}-4q)-4\varphi_2=0,$$

$$\psi_1=-{}^1/_2[\varphi_{0\tau}-\varphi_{0q}(\varphi_2+4q^2)],$$

$$\psi_3=-{}^1/_4[\varphi_{2\tau}-\varphi_{2q}(\varphi_2+4q^2)+4q\varphi_2].$$

In particular, the class of exact solutions (3.2) includes the solution

$$U=-{}^2/_3q^2Y^2+C_1q^{4/5}+{}^{10}/_3C_2q^{-1/5},$$

$$X=qY^2-{}^3/_4C_1q^{-1/5}-{}^5/_7C_2q^{-7/5}+C_3$$

$$V=-{}^4/_9q^3Y^3+C_1q^{4/5}Y-{}^{20}/_9C_2q^{4/5}Y,$$

$$(C_1,\ C_2,\ C_3=\text{const}).$$

§4. Consider a sharp-nosed body in a uniform supersonic flow. Upstream of the body there will be a compression shock, across which there must hold the dynamic compatibility conditions, which in the transonic approximation are [4]

$$2\frac{\partial X}{\partial\tau}+U_1+U+2\left(\frac{\partial X}{\partial Y}\right)^2=0,$$
$$(U-U_1)\frac{\partial X}{\partial Y}=V-V_1\qquad(4.1)$$

where $X = X(Y, \tau)$ represents the position of the shock, and U_1, V_1 are the components of the velocity vector upstream of the shock. We shall consider flows with an attached shock, represented by the parabola

$$X=q_0Y^2+2Dq_0Y.\qquad(4.2)$$

When the flow is non-steady, then $q = q_0(\tau)$ and $D = D_0(\tau)$ are functions of time. We shall construct the solution by means of the particular solutions (1.2). These solutions do not include enough arbitrariness to enable us to satisfy all boundary conditions at the shock and at the body surface for arbitrary body configuration. Therefore we shall construct a solution compatible with (4.1), (4.2), and shall take the streamline passing through the intersection of the coordinate axis and the shock as the body contour.

The conditions at the shock (4.1), at which $q = q_0$, impose the following boundary conditions on the functions φ_0, φ_1, χ_0, χ_1 for $q = q_0$

$$\varphi_2+8q_0^2=0,\quad 20q-\varphi_{2q}=0,$$
$${}^1/_2\varphi_2\varphi_{1q}-10q\varphi_1=0,\quad \varphi_1-\varphi_2\chi_{1q}=0,$$
$$\varphi_{0q}-4\chi_{1q}\chi_1-4q\chi_{0q}=0,\qquad(4.3)$$
$$\varphi_0-U_1-\chi_{0q}\varphi_2=0,\quad \psi_0-\chi_1\chi_{0q}\varphi_2=0.$$

For $q_0 = q(\tau)$ the compatibility conditions at the shock are

$$dq/d\tau=-(\varphi_2+4q^2)+{}^1/_2\varphi_2,$$

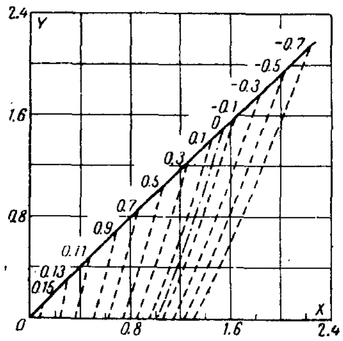
$$d\varphi_2/d\tau={}^1/_2\varphi_2(\varphi_{2q}-20q),$$

$$d\varphi_1/d\tau={}^1/_2\varphi_{1q}\varphi_2-6q\varphi_1-4\varphi_2\chi_1,$$

$$\varphi_1-\varphi_2\chi_{1q}=0,$$

$$d\varphi_0 / d\tau = {}^1/_2\varphi_2 (\varphi_{0q} - 4\chi_{1q}\chi_1 - 4q\chi_{0q}),$$

$$\varphi_0 - U_1 - \chi_{0q}\varphi_2 = 0, \quad \psi_0 - \chi_1\chi_{0q}\varphi_2 = 0.$$



Let us consider the steady case. From (4.3) it follows that if the flow upstream of the shock is uniform, then

$$\varphi_2 = -8q_0^2\check{\varphi}_2,$$

$$\check{\varphi}_2 = {}^{43}/_{121}\sqrt{12-11p} - {}^6/_{11}p + {}^{144}/_{121}.$$

Thus, a flow with a parabolic shock has the form

$$U = -8q_0^2\check{\varphi}_2 Y^2 - 16q_0^2 D\check{\varphi}_2 Y + \varphi_0 - 8q_0^2 D^2\varphi_2,$$

$$X = q_0 p Y^2 + 2Dq_0 Y p -$$

$$-\frac{1}{4q_0}\int_1^p \varphi_0 (2\check{\varphi}_2 - p^2)^{-1} dp + q_0 p D^2 - q_0 D^2,$$

$$V = {}^{32}/_3 q_0^3 [\check{\varphi}_{2p} (2\check{\varphi}_2 - p^2) + p\check{\varphi}_2] (Y+D)^3 - \qquad (4.4)$$

$$-8q_0 (Y+D) \{-{}^1/_3 (\check{\varphi}_{2p} - p)\varphi_0 + U_1 (2\check{\varphi}_2 - p^2)^{1/2} \times$$

$$\times [(12-11p)^{-1/2} + {}^2/_3 (12-11p)^{-3/4}]\} + 32q_0^3 D^2 Y,$$

$$\varphi_0 = -{}^1/_{11} U_1 |12 - 11p|^{1/4} (2\check{\varphi}_2 -$$

$$- p^2)^{-1/2} [3\sqrt{12-11p} + 8]. \qquad (4.4)$$

(cont'd)

The arbitrary constants C, D are determined from the given velocity U_1 and the angle of the shock at the point Y = 0. From (4.1) and (4.4) we get

$$C = \frac{U_1}{4},$$

$$D = \frac{1}{2q_0}\frac{dX}{dY}.$$

Here q_0 is a free parameter, which can be chosen so that the transition line will pass through the point X = 1, Y = 0. The figure represents the flow for the case $U_1 = 2.16$, $dX/dY = 1$. The dashed lines are lines of equal velocities and pressures, the solid line is the shock. The profile of the body in this case is close to the profile of a wedge with strong curvature of the walls near the transition line.

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